

ON THE LOCUS OF SMOOTH PLANE CURVES WITH A FIXED AUTOMORPHISM GROUP

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ABSTRACT. Let M_g be the moduli space of smooth, genus g curves over an algebraically closed field K of zero characteristic. Denote by $M_g(G)$ the subset of M_g of curves δ such that G (as a finite non-trivial group) is isomorphic to a subgroup of $\text{Aut}(\delta)$, the full automorphism group of δ , and let $\widetilde{M_g(G)}$ be the subset of curves δ such that $G \cong \text{Aut}(\delta)$. Now, for an integer $d \geq 4$, let M_g^{Pl} be the subset of M_g representing smooth, genus g plane curves of degree d (in such case, $g = (d-1)(d-2)/2$), and consider the sets $M_g^{Pl}(G) := M_g^{Pl} \cap M_g(G)$ and $\widetilde{M_g^{Pl}(G)} := \widetilde{M_g(G)} \cap M_g^{Pl}$.

In this paper, we study some aspects of the irreducibility of $\widetilde{M_g^{Pl}(G)}$ and its interrelation with the existence of “normal forms”, i.e. non-singular plane equations (depending on a set of parameters) such that a specialization of the parameters gives a certain non-singular plane model associated to the elements of $\widetilde{M_g^{Pl}(G)}$. In particular, we introduce the concept of being equation strongly irreducible (ES-Irreducible) for which the locus $\widetilde{M_g^{Pl}(G)}$ is represented by a single “normal form”. Henn, in [12], and Komiya-Kuribayashi, in [14], observed that $\widetilde{M_3^{Pl}(G)}$ is ES-Irreducible. In this paper we prove that this phenomena does not occur for any odd $d > 4$. More precisely, let $\mathbb{Z}/m\mathbb{Z}$ be the cyclic group of order m , we prove that $\widetilde{M_g^{Pl}(\mathbb{Z}/(d-1)\mathbb{Z})}$ is not ES-Irreducible for any odd integer $d \geq 5$, and the number of its irreducible components is at least two. Furthermore, we conclude the previous result when $d = 6$ for the locus $\widetilde{M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})}$.

Lastly, we prove the analogy of these statements when K is any algebraically closed field of positive characteristic p such that $p > (d-1)(d-2) + 1$.

1. INTRODUCTION

Let K be an algebraically closed field of zero characteristic and fix an integer $d \geq 4$. We consider, up to K -isomorphism, a projective non-singular curve δ of genus $g = (d-1)(d-2)/2$, and assume that δ has a non-singular plane model, i.e. $\delta \in M_g^{Pl}$.

It is well known that any $\delta \in M_g^{Pl}(G)$ corresponds to a set $\{C_\delta\}$ of non-singular plane models in $\mathbb{P}^2(K)$ such that any two of them are K -isomorphic through a projective transformation $P \in PGL_3(K)$ (where $PGL_N(K)$ is the classical projective linear group of $N \times N$ invertible matrices over K), and their automorphism groups are conjugate. More concretely, fixing a non-singular, degree d plane model C of δ whose defining equation is $F(X; Y; Z) = 0$. Then, $\text{Aut}(C)$ is a finite subgroup of $PGL_3(K)$, and also we have $\rho(G) \preceq \text{Aut}(C)$ for some injective representation $\rho : G \hookrightarrow PGL_3(K)$. Moreover, $\rho(G) = \text{Aut}(C)$ whenever $\delta \in \widetilde{M_g^{Pl}(G)}$. For another non-singular plane model C' of δ , there exists $P \in PGL_3(K)$ where C' is defined by $F(P(X, Y, Z)) = 0$, and $P^{-1}\rho(G)P \preceq \text{Aut}(C')$ (respectively, $P^{-1}\rho(G)P = \text{Aut}(C')$ if $\delta \in \widetilde{M_g^{Pl}(G)}$).

For an arbitrary, but a fixed degree d , classical and deep questions on the subject are: list the groups that appear as the exact automorphism groups of algebraic non-singular plane curves of degree d , and for each such group, determine the associated “normal forms”, i.e. a finite set of homogenous equations $\{N_{1,G}, \dots, N_{k,G}\}$ in X, Y, Z together with some parameters (under some restrictions) such that any specialization of a certain $N_{i,G}$ in K corresponds to a unique $\delta \in \widetilde{M_g^{Pl}(G)}$ (is the one that is associated to the non-singular plane model given by the specialization of the normal form $N_{i,G}$), and given any $\delta \in \widetilde{M_g^{Pl}(G)}$, there exists a unique i_δ and a specialization of the parameters in K for $N_{i_\delta,G}$ such that one obtains a plane non-singular model associated to

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δ . In particular, any specialization of the parameters of two distinct $N_{i,G}$ gives two non-singular plane models, which in turns relate to two non-isomorphic plane non-singular curves of $\widetilde{M_g^{Pl}(G)}$.

For $d = 4$, Henn in [12] and Komiya-Kuribayashi in [14], answered the above natural questions. See also Lorenzo's thesis [15] § 2.1 and § 2.2, in order to fix some minor details. It appears, for $d = 4$, the following phenomena: any element of $\widetilde{M_3^{Pl}(G)}$ has a non-singular plane model through some specialization of the parameters of a single normal form. If this phenomena appears for some g , we say that the locus $\widetilde{M_g^{Pl}(G)}$ is ES-Irreducible (see § 2 for the precise definition). This is a weaker condition than the irreducibility of this locus inside the moduli space M_g . In particular, it follows, by Henn [12] and Komiya-Kuribayashi [14], that the locus $\widetilde{M_3^{Pl}(G)}$ is always ES-Irreducible.

The motivation of this work is that we did not expect $\widetilde{M_g^{Pl}(G)}$ to be ES-Irreducible in general. In order to construct counter examples for which $\widetilde{M_g^{Pl}(G)}$ is not ES-Irreducible: we need first, a group G such that there exist at least two injective representations $\rho_i : G \hookrightarrow PGL_3(K)$ with $i = 1, 2$, which are not conjugate (i.e there is no transformation $P \in PGL_3(K)$ with $P^{-1}\rho_1(G)P = \rho_2(G)$, more details are included in § 2), and for the zoo of groups that could appear for non-singular plane curves [11], we consider G , a cyclic group of order m . Secondly, one needs to prove the existence of two non-singular plane curves with automorphism groups conjugate to $\rho_i(G)$ for each $i = 1, 2$.

The main results of the paper are that, the locus $\widetilde{M_g^{Pl}(\mathbb{Z}/(d-1)\mathbb{Z})}$ is not ES-irreducible for any odd degree $d \geq 5$, and it has at least two irreducible components (If $d=5$, we know by [2], that the only group G for which $\widetilde{M_6^{Pl}(G)}$ is not ES-Irreducible is $\mathbb{Z}/4\mathbb{Z}$). For d even, we prove in section § 5 that $\widetilde{M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})}$ is not ES-irreducible. It is to be noted that we may conjecture, by our work in [1], that the locus $\widetilde{M_g^{Pl}(\mathbb{Z}/m\mathbb{Z})}$ could not be ES-Irreducible only if m divides d or $d-1$ (this is true at least until degree 9 by [1]). Concerning positive characteristic, in the last section (§ 6) of this paper, we prove that the above examples of non-irreducible loci are also valid when K is an algebraically closed field of positive characteristic $p > 0$, provided that the characteristic p is big enough, once we fix the degree d .

The irreducibility of the loci $\widetilde{M_g^{Pl}(\mathbb{Z}/m)}$ seems to be very deep problem. In § 2, we give some insights that relate the above locus with subsets in classical loci of the moduli space. In particular, with the loci of curves of genus g with a prescribed cyclic Galois subcover. As an explicit example, we deal with the question for the locus $\widetilde{M_6^{Pl}(\mathbb{Z}/8)}$, which is ES-Irreducible, and is represented by a single normal form with only one paramater. In [1], we proved that $\widetilde{M_g^{Pl}(G)}$ is irreducible when G has an element of order $(d-1)^2$, $d(d-1)$, $d(d-2)$ or $d^2 - 3d + 3$, since this locus has only one element. In particular, we prove in [1] that $\widetilde{M_g^{Pl}(\mathbb{Z}/d(d-1))}$ and $\widetilde{M_g^{Pl}(\mathbb{Z}/(d-1)^2)}$ are irreducible.

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2. ON THE LOCI $\widetilde{M_g^{Pl}(G)}$ AND $\widetilde{M_g^{Pl}(G)}$.

Consider a non-singular projective curve δ of genus $g := \frac{(d-1)(d-2)}{2} \geq 2$ over K with a non-trivial finite subgroup G of $Aut(\delta)$. We always assume that δ admits a non-singular plane equation, and $\delta \in \widetilde{M_g^{Pl}(G)}$.

The linear systems g_d^2 are unique, up to multiplication by $P \in PGL_3(K)$ in $\mathbb{P}^2(K)$ [13, Lemma 11.28]. Therefore, we can take C as a non-singular plane model of δ , which is defined by a projective plane equation $F(X; Y; Z) = 0$, and $Aut(C)$ as a finite subgroup of $PGL_3(K)$ that fixes the equation F and is isomorphic to $Aut(\delta)$. Any other plane model of δ is given by $C_P : F(P(X; Y; Z)) = 0$ with $Aut(C_P) = P^{-1}Aut(C)P$ for some $P \in PGL_3(K)$, and C_P is K -equivalent or K -isomorphic to C . In particular, for $\delta \in \widetilde{M_g^{Pl}(G)}$, there exists $\rho : G \hookrightarrow PGL_3(K)$, where $\rho(G) \leq Aut(C)$ and $P^{-1}\rho(G)P \leq Aut(C_P)$.

We denote by $\rho(M_g^{Pl}(G))$ the locus of elements $\delta \in M_g^{Pl}(G)$ such that G acts on some plane model associated to δ as $P^{-1}\rho(G)P$ for some $P \in PGL_3(K)$, and similarly for $\rho(\widetilde{M_g^{Pl}(G)})$. Also, denote by A_G the quotient set $\{\rho : G \hookrightarrow PGL_3(K)\} / \sim$, where $\rho_1 \sim \rho_2$ if and only if $\exists P \in PGL_3(K)$ such that $\rho_1(G) = P^{-1}\rho_2(G)P$.

Clearly $M_g^{Pl}(G) = \cup_{[\rho] \in A_G} \rho(M_g^{Pl}(G))$, where $[\rho]$ denotes the class of ρ in A_G .

Lemma 2.1. *The locus $\widetilde{M_g^{Pl}(G)}$ is the disjoint union of $\rho(\widetilde{M_g^{Pl}(G)})$ where $[\rho]$ runs through A_G .*

Proof. It is clear, by definition, that $\widetilde{M_g^{Pl}(G)} = \cup_{[\rho] \in A_G} \rho(\widetilde{M_g^{Pl}(G)})$. Moreover, $\delta \in \rho_1(\widetilde{M_g^{Pl}(G)}) \cap \rho_2(\widetilde{M_g^{Pl}(G)})$ means that it has a plane model C such that $Aut(C) = P_1^{-1}\rho_1(G)P_1 = P_2^{-1}\rho_2(G)P_2$ for some $P_1, P_2 \in PGL_3(K)$. Therefore, $\rho_1 \sim \rho_2$. \square

Remark 2.2. *If $\delta \in \rho_1(M_g^{Pl}(G)) \cap \rho_2(M_g^{Pl}(G))$ with $[\rho_1] \neq [\rho_2] \in A_G$, and C is a plane model of δ , then $Aut(C) \leq PGL_3(K)$ should have two non-conjugate subgroups that are isomorphic to G . A detailed study of the work of Blichfeldt in [3] would give the list of G for which the decomposition $M_g^{Pl}(G) = \cup_{[\rho] \in A_G} \rho(M_g^{Pl}(G))$ may not be disjoint (if any).*

Fix a $[\rho] \in A_G$ then we can associate infinitely many non-singular plane models to $\delta \in \rho(M_g^{Pl}(G))$, which are K -isomorphic through a change of variables $P \in PGL_3(K)$. But we can consider only the models such that G is identified with $\rho(G) \leq PGL_3(K)$ for some ρ in $[\rho] \in A_G$ as, the full automorphism group. Under this restriction, δ can be associated with a non-empty family of non-singular models such that any two of them are isomorphic, through a projective transformation P that satisfies $P^{-1}\rho(G)P = \rho(G)$.

Recall that, it is a necessary condition, for a projective plane curve of degree d to be non-singular, that the defining equation of any model has degree $\geq d-1$ in each variable. For a non-zero monomial $cX^iY^jZ^k$, its exponent is defined to be $\max\{i, j, k\}$. For a homogeneous polynomial F , the core of F is defined as the sum of all terms of F with the greatest exponent. Now, we can assume, through a diagonal change of variables, that a non-singular plane model (whenever it exists) has only monic monomials in its core. Consequently, we reduce the situation to a set of K -isomorphic non-singular plane models $\{F_C(X; Y; Z) = 0\}$ associated to δ , where $\rho(G)$ leaves invariant the equation (being a subgroup of automorphisms of such model), and each term of the core of $F_C(X; Y; Z)$ is monic.

Lemma 2.3. *Consider G , a non-trivial finite group, and an injective representation $\rho : G \hookrightarrow PGL_3(K)$ of G such that $\rho(M_g^{Pl}(G))$ is non-empty. There exists a single normal form representing the locus $\rho(M_g^{Pl}(G))$, i.e. a homogenous polynomial $F_{\rho, G}(X; Y; Z) = 0$ of degree d in the variables X, Y and Z , which is endowed with some parameters as the coefficients of the lower order terms (together with some restrictions). More concretely, every specialization in K of the parameters of $F_{\rho, G}$ (with respect to the restrictions on the parameters) gives a non-singular plane model of an element $\delta \in \rho(M_g^{Pl}(G))$, and viceversa (any element $\delta \in \rho(M_g^{Pl}(G))$ corresponds to some specialization in K of the parameters of $F_{\rho, G}$ such that the resulting plane model of δ in $\mathbb{P}^2(K)$ is non-singular). A similar statement holds for the locus $\rho(\widetilde{M_g^{Pl}(G)})$. In such case, we call $F_{\rho, G, *}$ a single normal form. Moreover, these normal forms are unique up to change of the variables X, Y, Z by $P \in PGL_3(K)$.*

Proof. Let $\sigma \in G$ be an automorphism of maximal order $m > 1$, and choose an element ρ in $[\rho] \in A_G$ such that $\rho(\sigma)$ is diagonal of the form $diag(1, \xi_m^a, \xi_m^b)$ where $0 \leq a < b$, and ξ_m a primitive m -th root of unity in K . Following the same technique in [8] or in [1] (for a general discussion), we can associate to the set $\rho(M_g^{Pl}(\langle \sigma \rangle))$ a non-singular plane equation $F_{m, (a, b)}(X; Y; Z)$ with a certain set of parameters (under some algebraic restrictions in order to ensure the non-singularity). This is a “normal form” for $\rho(M_g^{Pl}(\langle \sigma \rangle))$, and it is also unique (up to K -equivalence) by construction. For example, if $0 < a < b < m$ and all the reference points $\{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$ satisfy the normal form $F_{m, (a, b)}(X; Y; Z) = 0$, we deduce that $F_{m, (a, b)}(X; Y; Z)$ is $X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X + \sum_{j=2}^{\lfloor \frac{d}{2} \rfloor} (X^{d-j}L_{j, X} + Y^{d-j}L_{j, Y} + Z^{d-j}L_{j, Z})$, where $L_{j, B}$ is a homogenous polynomial of degree j in the variables $\{X, Y, Z\} \setminus \{B\}$, and the parameters’ list is included as the coefficients of its monomials. The first three factors implies that $a \equiv (d-1)a + b \equiv (d-1)b \pmod{m}$. In particular, $m | d^2 - 3d + 3$. The defining equation $F_{m, (a, b)}$ in such situation, follows immediately by checking monomials’ invariance in each $L_{j, B}$. For example, rewrite $L_{j, X}$ as $\sum_{i=0}^j \beta_{j, i} Y^i Z^{j-i}$, where $\beta_{j, i}$ are parameters

in K . Then, $\beta_{j,i} = 0$ if $m \nmid ai + (j-i)b$, since $\text{diag}(1; \xi_m^a; \xi_m^b) \in \text{Aut}(C)$. Observe that, in order to obtain such $F_{m,(a,b)}$, we choose a model C for any $\delta \in \rho(M_g^{Pl}(G))$ that satisfies the condition $\rho(\sigma) = \text{diag}(1; \xi_m^a; \xi_m^b)$, and the monomials of the core are monic. Secondly, we impose that $\langle \sigma \rangle \subseteq \text{Aut}(C)$ to get the required unique expression.

Now, to move from $\rho(\langle \sigma \rangle)$ to $\rho(G)$, we assume a generator u_G of G , which does not belong to $\langle \sigma \rangle$, and then we use the fact that $\rho(u_G)$ retains invariant the defining equation $F_{m,(a,b)} = 0$ to obtain some specific algebraic relations between the parameters of $F_{m,(a,b)}$. Then, $F_{\rho,G}$ is obtained from $F_{m,(a,b)}$ by repeating the procedure for each such generator u_G and imposing the algebraic relations between the coefficients of the monomials (i.e. the parameters' list) of $F_{m,(a,b)}$. By a similar argument, we obtain $F_{\rho,G,*}$ from $F_{\rho,G}$. In fact, for a finite group H such that $\rho(G) \leq H \leq \text{PGL}_3(K)$, and for which there exists a non-singular plane curve of genus g whose automorphism group is isomorphic to H , we need to apply the process above for the generators of H that are not in $\rho(G)$. Therefore, we only need to consider a complement of certain algebraic constraints so that δ does not have a bigger automorphism group than H . □

Remark 2.4. *It could happen that two different specializations of $F_{\rho,G}$ in K give non-singular plane models that correspond to the same curve $\delta \in \rho(M_g^{Pl}(G))$. This happens if there exists a transformation P from one model to the other with $P^{-1}\rho(G)P = \rho(G)$, and $P^{-1}\rho(\langle \sigma \rangle)P = \rho(\langle \sigma \rangle)$. We can ensure that this phenomena will not occur by assuming more restrictions to the parameters of $F_{\rho,G}$, but we did not do this to our notion “normal form”. These further assumptions became recently explicit for the loci $\rho(M_3^{Pl}(G))$ through fixing the missing details in the tables of Henn [12], by Lorenzo [15]. We also investigated such restrictions particularly for the loci $\rho(M_6^{Pl}(\mathbb{Z}/8))$ and $\rho(\widetilde{M_6^{Pl}(\mathbb{Z}/8)})$ at the end of this section.*

It is difficult to determine the groups G and $[\rho] \in A_G$ such that $\rho(M_g^{Pl}(G))$ is non-empty for some fixed g . Henn [12] obtained this determination for $g = 3$, Badr-Bars [2] for $g = 6$, and for a general implementation of any degree $d \geq 5$, we refer to [1] (in which we formulate an algorithm to determine the ρ 's when G is cyclic).

Definition 2.5. Write $M_g^{Pl}(G)$ as $\cup_{[\rho] \in A_G} \rho(M_g^{Pl}(G))$, we define the number of the equation components of $M_g^{Pl}(G)$ to be the number of elements $[\rho] \in A_G$ such that $\rho(M_g^{Pl}(G))$ is not empty. We say that $M_g^{Pl}(G)$ is equation irreducible if it is not empty and $M_g^{Pl}(G) = \rho(M_g^{Pl}(G))$ for a certain $[\rho] \in A_G$. Similar notion arises for the locus $\widetilde{M_g^{Pl}(G)} = \cup_{[\rho] \in A_G} \widetilde{\rho(M_g^{Pl}(G))}$. We define the number of the strongly equation irreducible components of $\widetilde{M_g^{Pl}(G)}$ to be the number of the elements $[\rho] \in A_G$ such that $\rho(\widetilde{M_g^{Pl}(G)})$ is not empty. We say that $\widetilde{M_g^{Pl}(G)}$ is equation strongly irreducible (or simply, ES-irreducible) if it is not empty and $\widetilde{M_g^{Pl}(G)} = \rho(\widetilde{M_g^{Pl}(G)})$ for some $[\rho] \in A_G$.

Of course, if $\widetilde{M_g^{Pl}(G)}$ is not ES-irreducible then it is not irreducible and the number of the strongly irreducible equation components of $\widetilde{M_g^{Pl}(G)}$ is a lower bound for the number of its irreducible components.

In this language, we can formulate the main result in [12] as follows:

Theorem 2.6 (Henn, Komiya-Kuribayashi). *If G is a non-trivial group that appears as the full automorphism group of a non-singular plane curve of degree 4, then $\widetilde{M_3^P(G)}$ is ES-Irreducible.*

Remark 2.7. *Henn in [12], observed that $M_3^{Pl}(\mathbb{Z}/3)$ has already two irreducible equation components. But one of these components has always a bigger automorphism group namely, S_3 the symmetry group of order 3.*

To finish this section, we state some natural questions concerning the locus $\rho(M_g^{Pl}(G))$, and similar questions can be formulated for $\rho(\widetilde{M_g^{Pl}(G)})$ with different loci in the moduli space of genus g curves:

Question 2.8. *Is it true, for all the elements of $\rho(M_g^{Pl}(G))$, that the corresponding Galois covers $\delta \rightarrow \delta/G$ have a fixed ramification data?*

We believe that the answer to this question for $K = \mathbb{C}$ (i.e. Riemann surfaces) should be always true from the work of Breuer [4]. See Remark 4.4 for the explicit Galois subcover and the ramification data of the locus $\rho(M_6^{Pl}(\mathbb{Z}/4\mathbb{Z}))$, and §2.1 for the locus $\rho(M_6^{Pl}(\mathbb{Z}/8\mathbb{Z}))$.

Question 2.9. *Is $\rho(M_g^{Pl}(G))$ an irreducible set when G is a cyclic group?*

It is to be noted that when $K = \mathbb{C}$, Cornalba [7], for a cyclic group G of prime order, and Catanese [5], for general order, obtained that the locus of smooth projective genus g curves with a cyclic Galois subcover of a group that is isomorphic to G and a prescribed ramification is irreducible.

Concerning the irreducibility question, we prove in [1] that if G has an element of large order $(d-1)^2$, $d(d-1)$, $d(d-2)$ or $d^2 - 3d + 3$ then $\rho(M_g^{Pl}(G))$ has at most one element. Therefore, is irreducible. At §2.1, we deal with the irreducibility of the ES-Irreducible locus $M_6^{Pl}(\mathbb{Z}/8\mathbb{Z})$, where the single “normal form” has only one parameter.

Moreover, Catanese, Lönne and Perroni in [6, §2] define a topological invariant for the loci $M_g(G)$, which is trivial if it is irreducible.

Question 2.10. *Consider a non-trivial group G such that the set A_G is given by one element (see the next section for such groups). Is it a necessary condition that the topological invariant in [6, §2] is trivial in order to be irreducible? Is it true that the loci $M_g^{Pl}(G)$ are irreducible?*

2.1. The loci $M_6^{Pl}(\mathbb{Z}/8)$ and $M_6^{Pl}(\widetilde{\mathbb{Z}/8\mathbb{Z}})$.

Consider, in the moduli M_6 , an element δ that has a non-singular plane model with an effective action of the cyclic group of order 8. In other words, $\delta \in M_6^{Pl}(\mathbb{Z}/8\mathbb{Z})$. Following [1], [8] or the table in §4 of this note, one find that $M_6^{Pl}(\mathbb{Z}/8) = \rho(M_6^{Pl}(\mathbb{Z}/8))$ with $\rho(\mathbb{Z}/8\mathbb{Z}) = \langle \text{diag}(1, \xi_8, \xi_8^4) \rangle$, where ξ_8 is a 8-th primitive root of unity in K . Moreover, such loci has $X^5 + Y^4Z + XZ^4 + \beta X^3Z^2 = 0$ as a “normal form” with a parameter β that takes values in K such that $\beta \neq \pm 2$ (to ensure the non-singularity). Therefore, we can associate to δ a fixed plane non-singular model of the form $X^5 + Y^4Z + XZ^4 + \beta_\delta X^3Z^2 = 0$ for some $\beta_\delta \in K$ (there is no guarantee that β_δ is unique in K).

Now, let us compute all the non-singular plane models of the form $X^5 + Y^4Z + XZ^4 + \beta X^3Z^2 = 0$ that can be associated to the fixed curve δ . These models are obtained by a change of the variables $P \in PGL_3(K)$ such that $P^{-1} \langle \text{diag}(1, \xi_8, \xi_8^4) \rangle P = \langle \text{diag}(1, \xi_8, \xi_8^4) \rangle$, and the new model has a similar defining equation of the form $X^5 + Y^4Z + XZ^4 + \beta' X^3Z^2 = 0$.

Without any loss of generality, we can suppose that $P^{-1} \text{diag}(1, \xi_8, \xi_8^4) P = \text{diag}(1, \xi_8, \xi_8^4)$. Hence, in order to have the same eigenvalues which are pairwise distinct, we may assume that P is a diagonal matrix, say $P = \text{diag}(1, \lambda_2, \lambda_3)$. Therefore, we get an equation of the form: $X^5 + \lambda_2^4 \lambda_3 Y^4 Z + \lambda_3^4 X Z^4 + \beta_\delta \lambda_3^2 X^3 Z^2 = 0$. From which we must have $\lambda_2^4 \lambda_3 = \lambda_3^4 = 1$, thus λ_2^2 is 1 or -1. Consequently, we obtain a bijection map

$$\varphi : M_6^{Pl}(\mathbb{Z}/8\mathbb{Z}) \rightarrow \mathbb{A}^1(K) \setminus \{-2, 2\} / \sim$$

$$\delta \mapsto [\beta_\delta] = \{\beta_\delta, -\beta_\delta\}$$

where $a \sim b \Leftrightarrow b = a$ or $a = -b$. Furthermore, by our work in [2], we know that $X^5 + Y^4Z + XZ^4 + \beta X^3Z^2 = 0$ has a bigger automorphism group than $\mathbb{Z}/8\mathbb{Z}$ if and only if $\beta = 0$. Therefore, we have a bijection map

$$\tilde{\varphi} : M_6^{Pl}(\widetilde{\mathbb{Z}/8\mathbb{Z}}) \rightarrow \mathbb{A}^1(K) \setminus \{-2, 0, 2\} / \sim$$

$$\delta \mapsto [\beta_\delta] = \{\beta_\delta, -\beta_\delta\}.$$

The above sets are irreducible when K is the complex field.

On the other hand, if we consider the Galois cyclic cover of degree 8 that is given by the action of the automorphism of order 8 on $X^5 + Y^4Z + XZ^4 + \beta X^3Z^2 = 0$, we obtain that it ramifies at the points $[0 : 1 : 0]$ and $[0 : 0 : 1]$ with ramification index 8, as well as the four points $[1 : 0 : h]$, where $1 + h^4 + \beta h^2 = 0$ with ramification index 2, if $\beta \neq \pm 2$. That is, $M_6^{Pl}(\mathbb{Z}/8\mathbb{Z})$ is inside the locus of curves of the moduli space M_6 that have a cyclic Galois subcover of degree 8 to a genus zero curve, and also ramify at six points (two of them are with ramification index 8, and the other four points are with ramification index 4).

3. PRELIMINARIES ON AUTOMORPHISM ON PLANE CURVES

Consider a curve $\delta \in M_g^{Pl}$ whose $Aut(\delta)$ is non-trivial, and C is a fixed non-singular, degree d plane model of δ . By an abuse of notation (once and for all), we denote also by C a non-singular projective plane curve in \mathbb{P}^2 . Then, $Aut(C)$ is a finite subgroup of $PGL_3(K)$, and it satisfies one of the following situations (for more details, see Mitchell [16]):

- (1) fixes a point Q and a line L with $Q \notin L$ in $PGL_3(K)$,
- (2) fixes a triangle (i.e. a set of three non-concurrent lines),
- (3) $Aut(C)$ is conjugate to a representation inside $PGL_3(K)$ of one of the finite primitive groups namely, the Klein group $PSL(2, 7)$, the icosahedral group A_5 , the alternating group A_6 , the Hessian group $Hess_{216}$ or to one of its subgroups $Hess_{72}$ or $Hess_{36}$.

Recall that the exponent of a non-zero monomial $cX^iY^jZ^k$ is defined to be $\max\{i, j, k\}$. For a homogeneous polynomial F , the core of F is defined as the sum of all terms of F with the greatest exponent. Let C_0 be a smooth plane curve, a pair (C, \underline{G}) with $\underline{G} \leq Aut(C)$ is said to be a descendant of C_0 if C is defined by a homogeneous polynomial whose core is a defining polynomial of C_0 and \underline{G} acts on C_0 under a suitable change of the coordinate system.

Theorem 3.1 (Harui). *(see [11] §2) Let \underline{G} be a subgroup of $Aut(C)$ where C is a non-singular plane curve of degree $d \geq 4$. Then \underline{G} satisfies one of the following statements:*

- (1) \underline{G} fixes a point on C and then is cyclic.
- (2) \underline{G} fixes a point not lying on C and it satisfies a short exact sequence of the form

$$1 \rightarrow N \rightarrow \underline{G} \rightarrow G' \rightarrow 1,$$

where N a cyclic group of order dividing d and G' (which is a subgroup of $PGL_2(K)$) is conjugate to a cyclic group $\mathbb{Z}/m\mathbb{Z}$ of order m with $m \leq d - 1$, a Dihedral group D_{2m} of order $2m$ where $|N| = 1$ or $m|(d - 2)$, the alternating groups A_4, A_5 or the symmetry group S_4 .

- (3) \underline{G} is conjugate to a subgroup of $Aut(F_d)$, where F_d is the Fermat curve $X^d + Y^d + Z^d$. In particular, $|\underline{G}| \mid 6d^2$ and (C, \underline{G}) is a descendant of F_d .
- (4) \underline{G} is conjugate to a subgroup of $Aut(K_d)$, where K_d is the Klein curve $X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X$. Hence, $|H| \mid 3(d^2 - 3d + 3)$ and (C, H) is a descendant of K_d .
- (5) \underline{G} is conjugate to a finite primitive subgroup of $PGL_3(K)$ that are mentioned above.

The Hessian group: The representations of the Hessian group of order 216 $Hess_{216}$ inside $PGL_3(K)$ forms a unique set, up to conjugation (see Mitchell [16] page 217). A representation of $Hess_{216}$ in $PGL_3(K)$ is given by $Hess_{216} = \langle S, T, U, V \rangle$ where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad V = \frac{1}{\omega - \omega^2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Here ω is a primitive 3rd root of unity. Also, we consider the primitive Hessian subgroups of order 36, $Hess_{36}$ (one of them is $\langle S, T, V \rangle$), and the primitive subgroup of order 72, $Hess_{72} = \langle S, T, V, UVU^{-1} \rangle$.

For the above fixed representation, there are exactly three primitive subgroups of order 36 (see [9]), which are also normal in $Hess_{72}$. Moreover, the Hessian subgroup $Hess_{72}$ is normal in $Hess_{216}$. Furthermore, we recall, by Grove in [9, §23, p.25] and by Blichfeldt in [3] (see also [10, §1] for the statement of Blichfeldt's result of our interest) that any representation of $Hess_{216}$ corresponds geometrically to a certain subgroup fixing four triangles (having 18 elements), and the alternating group A_4 acting in such four triangles. Moreover, any representation of the primitive subgroups of order 36 or 72 is obtained by the group of 18 elements fixing the four triangles together with certain permutations on the four triangles (equivalently, with certain subgroups of A_4). On the other hand, it follows, by Blichfeldt (see [10, §1, on type (E),(F),(G)]), that such Hessian groups are represented in $PGL_3(K)$, up to conjugation, with respect to the representation described above. Therefore, any injective representation of $Hess_{36}$ or $Hess_{72}$ in $PGL_3(K)$ extends to an injective representation of $Hess_{216}$,

and moreover the three different subgroups of $Hess_{36}$ in any representation are conjugate to $\langle S, T, V \rangle$. Consequently, we conclude that the representations of $Hess_*$ with $*$ $\in \{36, 72, 216\}$ inside $PGL_3(K)$ form a unique set, up to conjugation.

Remark 3.2. In particular, for the Hessian groups $Hess_{216}$, $Hess_{72}$ and $Hess_{36}$, the locus $M_g^{Pl}(\widetilde{Hess_*})$, where $*$ $\in \{36, 72, 216\}$ is ES-Irreducible as long as it is not empty, since the set A_{Hess_*} is trivial (we follow the same notations of §2).

Our interest in investigating whether the locus $M_g^{Pl}(\widetilde{G})$ is ES-irreducible or not, and the classical result of Klein concerning the uniqueness (up to conjugation) of the finite subgroups inside $PGL_2(K)$, motivate us to ask the following question in group theory.

Question 3.3. Is it true that there exists \underline{G} , a non-cyclic finite subgroup of $PGL_3(K)$, such that the set $A_{\underline{G}}$ has at least two elements?

4. CYCLIC GROUPS IN SMOOTH PLANE CURVES OF DEGREE 5 AND $M_6^{Pl}(\mathbb{Z}/m\mathbb{Z})$.

We study non-singular plane curves $C : F(X; Y; Z) = 0$ of degree $d \geq 4$ such that $Aut(C)$ is non-trivial, up to K -isomorphism. That is, two of them are K -isomorphic if one transforms to the other through a change of variables $P \in PGL_3(K)$, and we denote by C_P the plane curve $F(P(X; Y; Z)) = 0$.

By a change of variables, we can suppose that the cyclic group of order m acting on a smooth plane curve of degree 5 is given in $PGL_3(K)$ by a diagonal matrix $diag(1; \xi_m^a, \xi_m^b)$, where ξ_m is an m -th primitive root of unity, and $0 \leq a < b < m$ are positive integers. We call this element by Type $m, (a, b)$.

Following the same proof of [8, §6.5] (or see [1], for a general treatment with an algorithm of computation for any degree d), we obtain a “normal form” associated to type $m, (a, b)$ corresponding to the loci $\rho(M_6^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ with $\rho(\mathbb{Z}/m) = \langle diag(1; \xi_m^a, \xi_m^b) \rangle$:

Type: $m, (a, b)$	$F_{m,(a,b)}(X; Y; Z)$
20, (4, 5)	$X^5 + Y^5 + XZ^4$
16, (1, 12)	$X^5 + Y^4Z + XZ^4$
15, (1, 11)	$X^5 + Y^4Z + YZ^4$
13, (1, 10)	$X^4Y + Y^4Z + Z^4X$
10, (2, 5)	$X^5 + Y^5 + XZ^4 + \beta_{2,0}X^3Z^2$
8, (1, 4)	$X^5 + Y^4Z + XZ^4 + \beta_{2,0}X^3Z^2$
5, (1, 2)	$X^5 + Y^5 + Z^5 + \beta_{3,1}X^2YZ^2 + \beta_{4,3}XY^3Z$
5, (0, 1)	$Z^5 + L_{5,Z}$
4, (1, 3)	$X^5 + X(Z^4 + Y^4 + \beta_{4,2}Y^2Z^2) + \beta_{2,1}X^3YZ$
4, (1, 2)	$X^5 + X(Z^4 + Y^4) + \beta_{2,0}X^3Z^2 + \beta_{3,2}X^2Y^2Z + \beta_{5,2}Y^2Z^3$
4, (0, 1)	$Z^4L_{1,Z} + L_{5,Z}$
3, (1, 2)	$X^5 + Y^4Z + YZ^4 + \beta_{2,1}X^3YZ + X^2(\beta_{3,0}Z^3 + \beta_{3,3}Y^3) + \beta_{4,2}XY^2Z^2$
2, (0, 1)	$Z^4L_{1,Z} + Z^2L_{3,Z} + L_{5,Z}$

where $L_{i,U}$ is a homogeneous polynomial of degree i that does not contain the variable U with parameters as the coefficients of the monomials, and $\beta_{i,j}$ are parameters that assume values in K . It remains to introduce the algebraic restrictions that should be imposed on the parameters $\beta_{i,j}$ so that the defining equation $F_{m,(a,b)}(X; Y; Z) = 0$ is non-singular. This will be omitted.

By the above table, we find that the locus $M_6^{Pl}(\mathbb{Z}/m\mathbb{Z})$ is not empty only for the values m which are included in the previous list. Moreover, we have $M_6^{Pl}(\mathbb{Z}/m\mathbb{Z}) = \rho(M_6^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ for $m \neq 4, 5$, where ρ is obtained such that $\rho(\mathbb{Z}/m\mathbb{Z}) = \langle diag(1, \xi_m^a, \xi_m^b) \rangle$. Thus, the corresponding loci $M_6^{Pl}(\widetilde{\mathbb{Z}/m\mathbb{Z}})$, where $m \neq 4, 5$, are ES-Irreducible provided that they are non-empty.

Now, we consider the remaining cases of the loci $M_6^{Pl}(\widetilde{\mathbb{Z}/m\mathbb{Z}})$ with $m = 4$ or 5 :

Obviously, the plane model of type 5, (1, 2) always have a bigger automorphism group by permuting X and Z . Therefore, there is at most one “normal form” that defines curves of degree 5 whose full automorphism

group is isomorphic to $\mathbb{Z}/5\mathbb{Z}$ (observe that the number of the conjugacy classes of representations of $\mathbb{Z}/5\mathbb{Z}$ in $PGL_3(K)$ is three). In particular, $M_6^{Pl}(\mathbb{Z}/5\mathbb{Z})$ is ES-Irreducible if it is non-empty. More precisely, $M_6^{Pl}(\mathbb{Z}/5\mathbb{Z}) = \rho(M_6^{Pl}(\mathbb{Z}/5\mathbb{Z}))$ with $\rho(\mathbb{Z}/5\mathbb{Z}) = \langle \text{diag}(1, 1, \xi_5) \rangle$ in this case.

On the other hand, for the cyclic groups of order 4, we have: Type 4, (1, 3) is not irreducible, since it is of the form $X \cdot G(X; Y; Z)$. Hence, it is singular, and will be out of the scope of this note. Therefore, we have $M_6^{Pl}(\mathbb{Z}/4\mathbb{Z}) = \rho_1(M_6^{Pl}(\mathbb{Z}/4\mathbb{Z})) \cup \rho_2(M_6^{Pl}(\mathbb{Z}/4\mathbb{Z}))$, where ρ_1 corresponds to Type 4, (0, 1) and ρ_2 to Type 4, (1, 2).

4.1. On type 4, (0, 1). Consider the non-singular plane curve defined by the equation

$$\tilde{C} : X^5 + Y^5 + Z^4X + \beta X^3Y^2 = 0,$$

where $\beta \neq 0$. This curve admits an automorphism of order 4 namely, $\sigma := [X; Y; \xi_4Z]$ that fixes pointwise the line $Z = 0$ (its axis) and the point $[0 : 0 : 1]$ off this line (its center). We call the elements of $PGL_3(K)$ that fix similar geometric constructions, homologies (for the element $\text{diag}(1; \xi_m^a; \xi_m^b) \in PGL_3(K)$ with $0 \leq a < b < m$, is an homology when $a = 0$). It follows, by §5 in [16], that $\text{Aut}(\tilde{C})$ should fix a point, a line or a triangle.

If $\text{Aut}(\tilde{C})$ fixes a triangle and neither a line nor a point is leaved invariant then, \tilde{C} is a descendant of the Fermat curve F_5 or the Klein curve K_5 (Harui [11], §5). But this is impossible, because $4 \nmid |\text{Aut}(F_5)| (= 150)$, and $4 \nmid |\text{Aut}(K_5)| (= 39)$. Therefore, $\text{Aut}(\tilde{C})$ should fix a line and a point off that line.

Now, the point $[0 : 0 : 1]$ is an inner Galois point of \tilde{C} , by Lemma 3.7 in [11]. Also, it is unique, by Yoshihara [18], §2, Theorem 4. Therefore, this point must be fixed by $\text{Aut}(\tilde{C})$. Moreover, the axis $Z = 0$ is also leaved invariant by Mitchell [16], §4. In particular, $\text{Aut}(\tilde{C})$ is cyclic by Lemma 11.44 in [13], and automorphisms of \tilde{C} are all diagonal of the form $[X; vY; tZ]$. This in turns implies that $v^5 = v^2 = t^4 = 1$. Hence, $v = 1$ and t is a 4-th root of unity. This shows that $\text{Aut}(\tilde{C})$ is cyclic of order 4.

Therefore, with the above argument we conclude the following result.

Proposition 4.1. *The locus set $\rho_1(\widetilde{M_6^{Pl}(\mathbb{Z}/4\mathbb{Z})})$ is non-empty.*

4.2. On type 4, (1, 2). Consider the non-singular plane curve defined by the equation

$$\tilde{\tilde{C}} : X^5 + X(Z^4 + Y^4) + \beta Y^2Z^3 = 0,$$

where $\beta \neq 0$. This curve admits a cyclic subgroup of automorphisms generated by $\tau := [X; \xi_4Y; \xi_4^2Z]$. For the same reason as above (i.e $4 \nmid |\text{Aut}(K_5)|, |\text{Aut}(F_5)|$), $\tilde{\tilde{C}}$ is not a descendant of the Fermat curve F_5 or the Klein curve K_5 . Moreover, $\text{Aut}(\tilde{\tilde{C}})$ is not conjugate to an icosahedral group A_5 (no elements of order 4), the Klein group $PSL(2, 7)$, the Hessian group $Hess_{216}$ or the alternating group A_6 (since by [11], Theorem 2.3, $|\text{Aut}(\tilde{\tilde{C}})| \leq 150$).

Now, we claim to prove that $\text{Aut}(\tilde{\tilde{C}})$ is also not conjugate to any of the Hessian subgroups namely, $Hess_{36}$ or $Hess_{72}$, and therefore it should fix a line and a point off that line: Let C be a non-singular plane curve of degree 5 such that $\text{Aut}(C)$ is conjugate, through $P \in PGL_3(K)$, to $Hess_*$ with $* \in \{36, 72, 216\}$. Then $\text{Aut}(C_P)$ is given by the usual presentation inside $PGL_3(K)$ of the above Hessian groups. In particular, $\text{Aut}(C_P)$ always has the following five elements: $[Z; Y; X]$, $[X; Z; Y]$, $[Y; X; Z]$, $[Y; Z; X]$ and $[X; \omega Y; \omega^2 Z]$, where ω is a primitive 3-rd root of unity. Because C_P is invariant by $[Z; Y; X]$, $[X; Z; Y]$, $[Y; X; Z]$ and $[Y; Z; X]$, then C_P must be of the form: $u(X^5 + Y^5 + Z^5) + a(X^4Z + X^4Y + Y^4X + Y^4Z + Z^4X + Z^4Y) + G(X; Y; Z)$, where $u, a \in K$, and $G(X; Y; Z)$ is a homogenous polynomial of degree at most three in each variable. Now, imposing the condition $[X; \omega Y; \omega^2 Z] \in \text{Aut}(C_P)$, we obtain that $u = 0$ and $a = 0$, a contradiction to non-singularity. Therefore, there is no non-singular, degree 5 plane curve whose automorphism group is conjugate to one of the Hessian groups. This proves our claim.

It follows, by the previous discussion, that $\text{Aut}(\tilde{\tilde{C}})$ should fix a line and a point off that line. Moreover, $\tau \in \text{Aut}(\tilde{\tilde{C}})$ is of the form $\text{diag}(1; a; b)$ such that $1, a, b$ (resp. $1, a^3, b^3$) are pairwise distinct then, automorphisms of $\tilde{\tilde{C}}$ are of the forms $\tau_1 := [X; vY + wZ; sY + tZ]$, $\tau_2 := [vX + wZ; Y; sX + tZ]$ or $\tau_3 := [vX + wY; sX + tY; Z]$ (because the fixed point is one of the reference points $[1 : 0 : 0]$, $[0 : 1 : 0]$ or $[0 : 0 : 1]$, and the fixed line is one of the reference lines $X = 0$, $Y = 0$ or $Z = 0$).

If $\tau_1 \in \text{Aut}(\tilde{C})$ then $s = 0 = w$ (Coefficient of Y^5 and Z^5), and we have the same conclusion, if τ_2 (resp. τ_3) $\in \text{Aut}(\tilde{C})$ from the coefficients of X^3Y^2 and Y^4Z (resp. Z^3X and YZ^4). Hence, automorphisms of \tilde{C} are all diagonal of the form $[X; vY; sZ]$. Moreover, $v^4 = s^4 = v^2s^3 = 1$, hence $v = \xi_4^r$, $s = \xi_4^{r'}$ with $(r, r') \in \{(0, 0), (2, 0), (1, 2), (3, 2)\}$. That is, $\text{Aut}(\tilde{C})$ is cyclic of order 4. Consequently, the following results follow.

Proposition 4.2. *The locus set $\rho_2(\widetilde{M_6^{Pl}(\mathbb{Z}/4\mathbb{Z})})$ is non-empty.*

Corollary 4.3. *The locus set $\widetilde{M_6^{Pl}(\mathbb{Z}/m\mathbb{Z})}$ is ES-Irreducible if and only if $m \neq 4$. If $m = 4$ then $\widetilde{M_6^{Pl}(\mathbb{Z}/m\mathbb{Z})}$ has exactly two irreducible equation components, and hence the number of its irreducible components is at least two.*

Remark 4.4. *Observe that for any element of $\rho_1(M_6^{Pl}(\mathbb{Z}/4\mathbb{Z}))$, the Galois cover of degree 4 corresponding to*

$$C_{\rho_1} := Z^4L_{1,Z} + L_{5,Z} = 0 \rightarrow C_{\rho_1} / < [X; Y; \xi_4Z] >$$

is ramified exactly at six points with ramification index 4. Indeed, the fixed points of σ^i for $i = 1, 2, 3, 4$ in $\mathbb{P}^2(K)$ are all the same set where $\sigma = \text{diag}(1, 1, \xi_4)$. Therefore, we only need to consider the ramification points of σ . In particular, the ramification index is always 4. Now, by the Hurwitz formula, we have $10 = 4(2g_0 - 2) + 3k$ where g_0 is the genus of $C_{\rho_1} / < [X, Y, \xi_4Z] >$. Hence, $g_0 = 0$ and $k = 6$. On the other hand, for any element of $\rho_2(\widetilde{M_6^{Pl}(\mathbb{Z}/4\mathbb{Z})})$, the Galois cover

$$C_{\rho_2} := X^5 + X(Z^4 + Y^4) + \beta_{2,0}X^3Z^2 + \beta_{3,2}X^2Y^2Z + \beta_{5,2}Y^2Z^3 = 0 \rightarrow C_{\rho_2} / < [X; \xi_4Y; \xi_4^2Z] >$$

is ramified at the points $[0 : 1 : 0], [0 : 0 : 1]$ with ramification index 4, and at the 4 points namely, $[1 : 0 : h]$ where $1 + h^4 + \beta_{2,0}h^2 = 0$ with ramification index 2 provided that $\beta_{2,0} \neq \pm 2$. We exclude the situation $\beta_{2,0} = \pm 2$ so that the defining equation is non-singular and geometrically irreducible.

Remark 4.5. *Given G , a non-trivial finite group, such that $\widetilde{M_6^{Pl}(G)}$ is non-empty. By a tedious work, one can show that $\widetilde{M_6^{Pl}(G)}$ is ES-Irreducible, except for the case $G \cong \mathbb{Z}/4\mathbb{Z}$ (for more details, we refer to [2]).*

Theorem 4.6. *Let $d \geq 5$ be an odd integer, and consider $g = (d-1)(d-2)/2$ as usual. Then $\widetilde{M_g^{Pl}(\mathbb{Z}/(d-1)\mathbb{Z})}$ is not ES-Irreducible, and it has at least two irreducible components.*

Proof. The above argument for concrete curves of Type 4, (0, 1) and Type 4, (1, 2) is valid for any odd degree $d \geq 5$, and the proof is quite similar. In other words, let \tilde{C} and $\tilde{\tilde{C}}$ be the non-singular plane curves of types $d-1, (0, 1)$ and $d-1, (1, 2)$ defined by the equations $X^d + Y^d + Z^{d-1}X + \beta X^{d-2}Y^2 = 0$, and $X^d + X(Z^{d-1} + Y^{d-1}) + \beta Y^2Z^{d-2} = 0$ respectively, where $\beta \neq 0$. Then, $\text{Aut}(\tilde{C})$ and $\text{Aut}(\tilde{\tilde{C}})$ are non-conjugate cyclic groups of order $d-1$, and are generated by $[X; Y; \xi_{d-1}Z]$ and $[X; \xi_{d-1}Y; \xi_{d-1}^2Z]$ respectively. Therefore, they belong to two different $[\rho]'$ s.

On type $d-1, (0, 1)$: With a homology of order $d-1 \geq 4$ inside $\text{Aut}(\tilde{C})$, we conclude that $\text{Aut}(\tilde{C})$ fixes a point, a line or a triangle (See [16], §5). Furthermore, the center $[0 : 0 : 1]$ of this homology is an inner Galois point, by Lemma 3.7 in [11]. Also, it is unique, by Theorem 4 in [18]. Therefore, it should be fixed by $\text{Aut}(\tilde{C})$, and also the axis $Z = 0$ is leaved invariant, by Theorem 4 in [16]. Hence, $\text{Aut}(\tilde{C})$ is cyclic, by Lemma 11.44 in [13], and automorphisms of \tilde{C} are of the form $\text{diag}(1; v; t)$ such that $v^d = t^{d-1} = v = 1$. That is, $|\text{Aut}(\tilde{C})| = d-1$.

On type $d-1, (1, 2)$: First, we prove that $\text{Aut}(\tilde{\tilde{C}})$ fixes a line and a point off this line: We consider the case $d \geq 7$ (For $d = 5$, we refer to the previous results). The alternating group A_6 has no elements of order $d-1 \geq 6$. The Klein group $PSL(2, 7)$, which is the only simple group of order 168, has no elements of order ≥ 8 , and also there are no elements of order 6 inside (for more details, we refer to [17]). Therefore, the primitive groups A_5, A_6 , and $PSL(2, 7)$ do not appear as the full automorphism group. Moreover, elements inside the Hessian group $Hess_{216} \cong \text{SmallGroup}(216, 153)$ have orders 1, 2, 3, 4 and 6. Then $Hess_*$ with $*$ $\in \{36, 72, 216\}$

do not appear as the full automorphism group, except possibly for $d = 7$. On the other hand, $d-1 \nmid 3(d^2-3d+3)$ hence \tilde{C} is not a descendant of the Klein curve K_d . Furthermore, \tilde{C} is not a descendant of the Fermat curve F_d , because $d-1 \nmid 6d^2$ (except for $d = 7$).

Finally, it remains to deal with the case $d = 7$ for the Hessian groups or for being a Fermat's descendant. By the same line of argument as for the claim of Type 4, (1, 2), we can show that non of the Hessian groups could appear for a non-singular, degree 7, plane curve. Also, the automorphisms of the Fermat curve F_7 are of the forms $[X; \xi_7^a Y; \xi_7^b Z]$, $[\xi_7^b Z; \xi_7^a Y; X]$, $[X; \xi_7^b Z; \xi_7^a Y]$, $[\xi_7^a Y; X; \xi_7^b Z]$, $[\xi_7^a Y; \xi_7^b Z; X]$, $[\xi_7^b Z; X; \xi_7^a Y]$. One can easily verify that non of them has order 6. Consequently, we exclude the possibility of being a Fermat's descendant.

Now, the full automorphism group should fix a line and a point off this line. Thus automorphisms of \tilde{C} have the forms $[X; vY + wZ; sY + tZ]$, $[vX + wZ; Y; sX + tZ]$ or $[vX + wY; sX + tY; Z]$, since $[X; \xi_{d-1} Y; \xi_{d-1}^2 Z] \in \text{Aut}(\tilde{C})$.

If $[X; vY + wZ; sY + tZ] \in \text{Aut}(\tilde{C})$ then $s = 0 = w$ (Coefficient of Y^d and Z^d), and the same conclusion follows if $[vX + wZ; Y; sX + tZ]$ (resp. $[vX + wY; sX + tY; Z]$) from the coefficients of $X^{d-2}Y^2$ and $Y^{d-1}Z$ (resp. $Z^{d-2}X^2$ and YZ^{d-1}). Hence, automorphisms of \tilde{C} are all diagonal of the form $\text{diag}(1; v; s)$. Moreover, $v^{d-1} = s^{d-1} = v^2 s^{d-2} = 1$ that is, $v = \xi_{d-1}^r$ and $s = \xi_{d-1}^{r'}$ such that $d-1 \mid 2r - r'$. Therefore, automorphisms of \tilde{C} are $[X; \xi_{d-1}^r Y; \xi_{d-1}^{2r} Z]$ with $r \in \{0, 1, \dots, d-2\}$. Hence, $\text{Aut}(\tilde{C})$ is cyclic of order $d-1$, which was to be shown. \square

5. ON THE LOCUS $\widetilde{M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})}$.

By a similar argument as the degree 5 case, we obtain the following “normal forms” for $\rho(M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z}))$, (see the full table for degree 6 in [1]):

Type: $m, (a, b)$	$F_{m,(a,b)}(X; Y; Z)$
3, (0, 1)	$Z^6 + Z^3 L_{3,Z} + L_{6,Z}$
3, (1, 2)	$X^5 Y + Y^5 Z + Z^5 X + \mu_1 Z^2 X^4 + \mu_2 X^2 Y^4 + \mu_3 Y^2 Z^4 + \alpha_1 X^3 Y^2 Z + \alpha_2 X Y^3 Z^2 + \alpha_3 X^2 Y Z^3$

where μ_i, α_i are parameters that take values in K so that the associated models of the respective loci $\rho(M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z}))$ are non-singular.

5.1. On type 3, (1, 2).

Proposition 5.1. *Let $\delta \in M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})$ such that δ admits a non-singular plane model \tilde{C} of the form*

$$X^5 Y + Y^5 Z + Z^5 X + \mu_1 Z^2 X^4 + \mu_2 X^2 Y^4 + \mu_3 Y^2 Z^4 + \alpha_1 X^3 Y^2 Z + \alpha_2 X Y^3 Z^2 + \alpha_3 X^2 Y Z^3 = 0.$$

Then, $\text{Aut}(\tilde{C})$ either fixes a line and a point off that line or it fixes a triangle.

Proof. It suffices to show that $\text{Aut}(\tilde{C})$ is not conjugate to any of the finite primitive groups inside $PGL_3(K)$ namely, the Klein group $PSL(2, 7)$, the icosahedral group A_5 , the alternating group A_6 , the Hessian group $Hess_{216}$ or to any of its subgroups $Hess_{72}$ or $Hess_{36}$, and the result follows by Mitchell [16].

Let $\tau \in \text{Aut}(\tilde{C})$ be an element of order 2 such that $\tau\sigma\tau = \sigma^{-1}$, where $\sigma := [X; \omega Y; \omega^2 Z]$ then τ has one of the forms $[X; \beta Z; \beta^{-1} Y]$, $[\beta Y; \beta^{-1} X; Z]$ or $[\beta Z; Y; \beta^{-1} X]$. But non of these transformations retains \tilde{C} , hence $\text{Aut}(\tilde{C})$ does not contain an S_3 as a subgroup. Consequently, $\text{Aut}(\tilde{C})$ is not conjugate to A_5 or A_6 . Moreover, it is well known that $PSL(2, 7)$ contains an octahedral group of order 24 (but not an isocahedral group of order 60), and since all elements of order 3 in $PSL(2, 7)$ are conjugate (for more details, we refer to [17]). Then, by the same argument as before, we conclude that $\text{Aut}(\tilde{C})$ is not conjugate to $PSL(2, 7)$. Lastly, assume that $\text{Aut}(\tilde{C})$ is conjugate, through a transformation P , to one of the Hessian groups say, $Hess_*$. Then, we can consider $P^{-1}SP = \lambda S$, because we did not fix the plane model for a curve whose automorphism group is $Hess_*$. In particular, P should be of the form $[Y; \gamma Z; \beta X]$, $[Z; \gamma X; \beta Y]$ or $[X; \gamma Y; \beta Z]$, but non of them transform \tilde{C} to

\tilde{C}_P with $\{[X; Z; Y], [Y; X; Z], [Z; Y; X]\} \subseteq \text{Aut}(\tilde{C}_P)$. Therefore, $\text{Aut}(\tilde{C})$ is not conjugate to any of the Hessian groups, and we have done. \square

Notations. Let $\Gamma := \{(\beta'_1, \beta'_2, \beta'_3) \in K^* \times K^* \times K^* : \Upsilon_1 = 1, \Upsilon_2 = \Upsilon_3 = \omega^2 \Upsilon_4\}$, where

$$\begin{aligned}\Upsilon_1(\beta'_1, \beta'_2, \beta'_3) &:= \beta'_3 \beta'_2{}^5 + (\beta'_1 \beta'_3{}^3 + 1) \beta'_2 + \beta'_3{}^5, \\ \Upsilon_2(\beta'_1, \beta'_2, \beta'_3) &:= \lambda'^2 \left((5\lambda'^3 + 1) \beta'_3 \beta'_2{}^5 + (5\lambda'^6 + \lambda'^3 + (2\lambda'^6 + \lambda'^3 + 3) \beta'_1 \beta'_3{}^3) \beta'_2 + (\lambda'^6 + 5) \beta'_3{}^5 \right), \\ \Upsilon_3(\beta'_1, \beta'_2, \beta'_3) &:= \lambda'^5 \left((\lambda'^6 + 5) \beta'_3 \beta'_2{}^5 + (5\lambda'^3 + (3\lambda'^6 + 2\lambda'^3 + 1) \beta'_1 \beta'_3{}^3 + 1) \beta'_2 + \lambda'^3 (5\lambda'^3 + 1) \beta'_3{}^5 \right), \\ \Upsilon_4(\beta'_1, \beta'_2, \beta'_3) &:= \lambda' \left(\lambda'^4 (5\lambda'^3 + 1) \beta'_3 \beta'_2{}^5 + \lambda' (\lambda'^6 + (\lambda'^6 + 3\lambda'^3 + 2) \beta'_1 \beta'_3{}^3 + 5) \beta'_2 + \lambda' (5\lambda'^3 + 1) \beta'_3{}^5 \right),\end{aligned}$$

and $\lambda'^3 = \xi_6^2 = \omega$. Also, we define Γ_1 to be the set of all values that appear in the first coordinate of elements of Γ , which (by a computation) is a finite subset of K^* . Now, we state and prove the main result for this section:

Theorem 5.2. *Consider an element $\delta \in M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})$ that has a non-singular plane model \tilde{C} of the form $\tilde{C} : X^5Y + Y^5Z + Z^5X + \alpha_3 X^2YZ^3 = 0$ with $\alpha_3 \neq 0$, and assume for simplicity that $\alpha_3 \notin \Gamma_1$. The full automorphism group of such δ is cyclic of order 3, and is generated by the transformation $\sigma : (x; y; z) \mapsto (x; \omega y; \omega^2 z)$.*

Proof. It follows, by Proposition 5.1, that $\text{Aut}(\tilde{C})$ either fixes a line and a point off that line or it fixes a triangle. We treat each of these two cases.

- (1) If $\text{Aut}(\tilde{C})$ fixes a line L and a point P off this line, then L must be one of the reference lines $B = 0$, where $B \in \{X, Y, Z\}$, and P is one of the reference points namely, $[1 : 0 : 0]$, $[0 : 1 : 0]$ or $[0 : 0 : 1]$ (being $\sigma \in \text{Aut}(\tilde{C})$). Consequently, $\text{Aut}(\tilde{C})$ is cyclic, since all the reference points lie on \tilde{C} . Also, automorphisms of \tilde{C} are of the forms

$$\tau_1 := [X; vY + wZ; sY + tZ], \tau_2 := [vX + wZ; Y; sX + tZ] \text{ or } \tau_3 := [vX + wY; sX + tY; Z]$$

For τ_1 to be in $\text{Aut}(\tilde{C})$, we must have $w = 0 = s$ (coefficients of X^5Z and XY^5), and similarly, for τ_2 (resp. τ_3) through the coefficients of Y^5X and Z^6 (resp. YZ^5 and X^5Z). That is, elements of $\text{Aut}(\tilde{C})$ are all diagonal of the form $\text{diag}(1; v; t)$ such that $tv^4 = 1 = t^3$ and $t^5 = v$. Thus, $t = \xi_3^a$ and $v = \xi_3^{2a}$, where ξ_3 is a primitive 3-rd root of unity, and hence, $|\text{Aut}(\tilde{C})| = 3$.

- (2) If $\text{Aut}(\tilde{C})$ fixes a triangle and there exist neither a line nor a point leaved invariant, then by Harui [11], \tilde{C} is a descendant of the Fermat curve $F_6 : X^6 + Y^6 + Z^6$ or the Klein curve $K_6 : X^5Y + Y^5Z + Z^5X$. Hence, $\text{Aut}(\tilde{C})$ is conjugate to a subgroup of $\text{Aut}(F_6) = \langle [\xi_6 X; Y; Z], [X; \xi_6 Y; Z], [Y; Z; X], [X; Z; Y] \rangle$ or to a subgroup of $\text{Aut}(K_6) = \langle [Z; X; Y], [X; \xi_{21} Y; \xi_{21}^{-4} Z] \rangle$.

- Suppose first that $\text{Aut}(\tilde{C})$ is conjugate (through P) to a subgroup of $\text{Aut}(F_6)$. Then, it suffices to assume that $P^{-1}SP \in \{S, [Y; Z; X], [Y; \xi_6 Z; X], [Y; \xi_6^2 Z; X]\}$, since any element of order 3 in $\text{Aut}(F_6)$, which is not a homology, is conjugate to one of those inside $\text{Aut}(F_6)$. Now, if $P^{-1}SP = S$ then $P \in \text{PGL}_3(K)$ is of the form $[Y; \gamma Z; \beta X]$, $[Z; \gamma X; \beta Y]$ or $[X; \gamma Y; \beta Z]$, but non of them transforms \tilde{C} to \tilde{C}_P with core $X^6 + Y^6 + Z^6$, a contradiction. Furthermore, if $P^{-1}SP = [Y; Z; X]$

(resp. $= [Y; \xi_6 Z; X]$ or $= [Y; \xi_6^2 Z; X]$), then P has the form $\begin{pmatrix} \lambda & 1 & \lambda^2 \\ \omega \lambda \beta_2 & \beta_2 & \omega^2 \lambda^2 \beta_2 \\ \omega^2 \lambda \beta_3 & \beta_3 & \lambda^2 \omega \beta_3 \end{pmatrix}$, where $\lambda^3 = 1$

(resp. $\lambda^3 = \xi_6$ or $\lambda^3 = \xi_6^2$). We thus get \tilde{C}_P of the form $\Upsilon_1(\alpha_3, \beta_2, \beta_3)(\lambda^6 \omega X^6 + Y^6 + \lambda^{12} \omega^2 Z^6) + \dots$. In particular, $\Upsilon_1(\alpha_3, \beta_2, \beta_3) = 1$, $\lambda^3 = \xi_6^2$ and $[Y; \xi_6^2 Z; X] \in \text{Aut}(\tilde{C}_P)$. Consequently, \tilde{C}_P should be of the form $X^6 + Y^6 + Z^6 + (\Upsilon_2(\alpha_3, \beta_2, \beta_3) X^5Y + \Upsilon_4(\alpha_3, \beta_2, \beta_3) Y^5Z + \Upsilon_3(\alpha_3, \beta_2, \beta_3) XZ^5) + \dots$, which must be reduced to the form $X^6 + Y^6 + Z^6 + \Upsilon_2(\alpha_3, \beta_2, \beta_3)(X^5Y + \omega Y^5Z + XZ^5) + \dots$, because $[Y; \xi_6^2 Z; X] \in \text{Aut}(\tilde{C}_P)$. This could happen only if $\alpha_3 \in \Gamma_1$, which is not possible by the assumptions on α_3 . Therefore, \tilde{C} is not a descendant of the Fermat curve F_6 .

- Secondly, suppose that \tilde{C} is a descendant of the Klein curve K_6 . This should happen through a change of the variables $P \in \text{PGL}_3(K)$ such that $\tilde{C}_P : X^5Y + Y^5Z + Z^5X + \text{lower terms}$. We claim to show that $P^{-1}SP = \lambda S$ for some $\lambda \in K^*$. Indeed, elements of order 3 inside $\text{Aut}(K_6)$, which

are not homologies, are $S, S^{-1}, [\xi_{21}^a Y; \xi_{21}^{-4a} Z; X]$ and $[\xi_{21}^{-4a} Z; X; \xi_{21}^a Y]$, and it is enough to consider the situation $P^{-1}SP \in \{S, S^{-1}, [\xi_{21}^a Y; \xi_{21}^{-4a} Z; X], [\xi_{21}^{-4a} Z; X; \xi_{21}^a Y]\}$ with $a = 0, 1, 2$, because any other value is conjugate inside $\text{Aut}(K_6)$ to one of these transformations.

If $P^{-1}SP = \lambda S^{-1}$ then P fixes one of the variables and permutes the others. Hence, the resulting core is different from $X^5Y + Y^5Z + Z^5X$, a contradiction.

If $P^{-1}SP = \lambda[\xi_{21}^a Y; \xi_{21}^{-4a} Z; X]$ (resp. $[\xi_{21}^{-4a} Z; X; \xi_{21}^a Y]$) then P has the form

$$\begin{pmatrix} \lambda \xi_{21}^{-a} & 1 & \lambda^2 \xi_{21}^{-a} \\ \lambda \xi_{21}^{-a} \omega \beta_2 & \beta_2 & \lambda^2 \xi_{21}^{-a} \omega^2 \beta_2 \\ \lambda \xi_{21}^{-a} \omega^2 \beta_3 & \beta_3 & \lambda^2 \xi_{21}^{-a} \omega \beta_3 \end{pmatrix} \quad (\text{resp.} \quad \begin{pmatrix} \lambda^2 \xi_{21}^{-18a} & 1 & \lambda \xi_{21}^{-a} \\ \lambda^2 \xi_{21}^{-18a} \omega^2 \beta_2 & \beta_2 & \lambda \xi_{21}^{-a} \omega \beta_2 \\ \lambda^2 \xi_{21}^{-18a} \omega \beta_3 & \beta_3 & \lambda \xi_{21}^{-a} \omega^2 \beta_3 \end{pmatrix})$$

where $\lambda^3 = \xi_{21}^{-3a}$. For both transformations, we must have $\beta_3 \beta_2^5 + (\alpha_3 \beta_3^3 + 1) \beta_2 + \beta_3^5 = 0$ so that X^6, Y^6, Z^6 do not appear. Therefore, by imposing the condition X^5Z, XY^5 and YZ^5 do not appear as well, we get $\alpha_3 = 0$, which is already excluded. Consequently, $P^{-1}SP = \lambda S$, and we proved the claim. Now, P has one of the forms $[Y; \gamma Z; \beta X]$, $[Z; \gamma X; \beta Y]$ or $[X; \gamma Y; \beta Z]$. Therefore, \tilde{C}_P is defined by an equation of the form $\lambda_0(X^5Y + Y^5Z + Z^5X) + \lambda_1 G(X; Y; Z)$, where $G(X; Y; Z)$ is one of the monomials X^2YZ^3, Y^2ZX^3 , or Z^2XY^3 . In particular, $[\mu_1 Z; X; \mu_2 Y] \notin \text{Aut}(\tilde{C}_P)$, and $\text{Aut}(\tilde{C}_P) \preceq \tau := [X; \xi_{21} Y; \xi_{21}^{-4} Z] \succ$. Moreover, $\tau^r \in \text{Aut}(\tilde{C}_P)$ if and only if $7|r$. Hence, $\text{Aut}(\tilde{C})$ is cyclic of order 3.

This completes the proof. \square

5.2. On type 3, (0, 1).

Proposition 5.3. *If $\delta \in M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})$ has a non-singular plane model \tilde{C} of the form $Z^6 + Z^3 L_{3,Z} + L_{6,Z} = 0$, then $\text{Aut}(\tilde{C})$ is either conjugate to the Hessian group Hess_{216} or it leaves invariant a point, a line or a triangle.*

Proof. The result is an immediate consequence, since $\text{Aut}(\tilde{C})$ contains a homology (i.e. leaves invariant a line pointwise and a point off this line) of period 3 namely, $\sigma' := [X; Y; \omega Z]$, and Hess_{216} is the only multiplicative group that contains such homologies and does not leave invariant a point, a line or a triangle (See Theorem 9, [16]). \square

Now, we can prove our main result for this section.

Theorem 5.4. *The automorphisms group of an element $\delta \in M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})$ with a non-singular plane model \tilde{C} of the form $Z^6 + X^5Y + XY^5 + \alpha_3 Z^3 X^3 = 0$ such that $\alpha_3 \neq 0$ is cyclic of order 3, and is generated by the automorphism $\sigma' : (x; y; z) \mapsto (x; y; \omega z)$.*

Proof. Suppose that $\text{Aut}(\tilde{C})$ is conjugate, through a transformation P , to the Hessian group Hess_{216} . Then, we can assume, without loss of generality, that $P^{-1}\sigma'P = \lambda\sigma'$ for some $\lambda \in K^*$. Hence, $P = [\alpha_1 X + \alpha_2 Y; \beta_1 X + \beta_2 Y; Z]$ and clearly, $\{[Z; Y; X], [X; Z; Y]\} \not\subseteq \text{Aut}(\tilde{C}_P)$, a contradiction. Therefore, we deduce, by Proposition 5.3, that $\text{Aut}(\tilde{C})$ should fix a point, a line or a triangle.

In what follows, we treat each case.

- (1) If $\text{Aut}(\tilde{C})$ fixes a line and a point off that line, and if \tilde{C} admits a bigger non cyclic automorphism group, then $\text{Aut}(\tilde{C})$ satisfies a short exact sequence of the form $1 \rightarrow \mathbb{Z}/3\mathbb{Z} \rightarrow \text{Aut}(\tilde{C}) \rightarrow G' \rightarrow 1$, where G' is conjugate to $\mathbb{Z}/m\mathbb{Z}$ ($m = 2, 3$ or 4), D_{2m} ($m = 2$ or 4), A_4, S_4 or A_5 .

If G' is conjugate to $\mathbb{Z}/3\mathbb{Z}, A_4, S_4$ or A_5 , then there exists, by Sylow's theorem, a subgroup H of automorphisms of \tilde{C} of order 9. In particular, H is conjugate to $\mathbb{Z}/9\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, but both cases do not occur. Indeed, if H is conjugate to $\mathbb{Z}/9\mathbb{Z}$ then $\text{Aut}(\tilde{C})$ has an element of order 9, which is not possible because $9 \nmid d-1, d, (d-1)^2, d(d-2), d(d-1), d^2 - 3d + 3$ with $d = 6$ (for more details, we refer to [1]). Moreover, if H is conjugate to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ then there exists $\tau \in \text{Aut}(\tilde{C})$ of order 3 such that $\tau\sigma' = \sigma'\tau$. Hence, $\tau = [vX + wY; sX + tY; Z]$, and comparing the coefficients of Z^3Y^3 and X^6 in \tilde{C}_τ , we get $w = 0 = s$ and $v^5t = vt^5 = v^3 = 1$. Thus $\tau \in \langle \sigma' \rangle$, a contradiction.

By a similar argument, we exclude the cases $\mathbb{Z}/4\mathbb{Z}$ and D_{2m} , because for each $SmallGroup(6m, ID)$, there must be an element τ of order 2 or 4 which commutes with σ' .

Finally, if G' is conjugate to $\mathbb{Z}/2\mathbb{Z}$ then there exists an element τ of order 2 such that $\tau\sigma'\tau = \sigma'^{-1}$ and one can easily verify that such an element does not exist. This follows immediately because $\tau = \tau^{-1}$ (being of order 2), and since σ' and σ'^{-1} are not conjugate in $PGL_3(K)$.

We conclude that $Aut(\tilde{C})$ should be cyclic (in particular, is commutative). Hence, it can not be of order > 3 (otherwise; there must be an element $\tau \in Aut(\tilde{C})$ of order > 3 which commutes with σ' , and by a previous argument, such elements do not exist).

- (2) If $Aut(\tilde{C})$ fixed a triangle and neither a point nor a line is fixed, then it follows, by Harui [11], that \tilde{C} is a descendant of the Fermat curve F_6 or the Klein curve K_6 . The last case does not happen, because $Aut(K_6)$ does not have elements of order 3 whose Jordan form is the the same as σ' (i.e a homology). Now, suppose that \tilde{C} is a descendant of F_6 that is, \tilde{C} can be transformed (through P) into a curve \tilde{C}_P whose core is $X^6 + Y^6 + Z^6$. Then, $P = [\alpha_1 X + \alpha_2 Y; \beta_1 X + \beta_2 Y; Z]$, since there are only two sets of homologies in $Aut(F_6)$ of order 3 namely, $\{[\omega X; Y; Z], [X; \omega Y; Z], [X; Y; \omega Z]\}$ and $\{[\omega^2 X; Y; Z], [X; \omega^2 Y; Z], [X; Y; \omega^2 Z]\}$ (recall that the two sets are not conjugate in $PGL_3(K)$). Also, elements of the first set are all conjugate inside $Aut(F_6)$ to $[X; Y; \omega Z]$. So it suffices to consider the situation $P^{-1}\sigma P = \lambda\sigma$. Now, \tilde{C}_P has the form

$$\mu_0 X^6 + \mu_1 Y^6 + Z^6 + \alpha_3(\alpha_1 X + \alpha_2 Y)^3 Z^3 + \mu_2 X^5 Y + \mu_3 X^4 Y^2 + \mu_4 X^3 Y^3 + \mu_5 X^2 Y^4 + \mu_6 X Y^5,$$

where $\mu_0 := \alpha_1 \beta_1 (\alpha_1^4 + \beta_1^4) (= 1)$ and $\mu_1 := \alpha_2 \beta_2 (\alpha_2^4 + \beta_2^4) (= 1)$. In particular, $(\alpha_1 \beta_1)(\alpha_2 \beta_2) \neq 0$ therefore, $[X; vZ; wY]$, $[vZ; wY; X]$, $[wY; vZ; X]$, and $[vZ; X; wY] \notin Aut(\tilde{C}_P)$, because of the monomial $XY^2 Z^3$. Moreover, $[wY; X; vZ] \in Aut(\tilde{C}_P)$ only if $\alpha_1 = \alpha_2$ and $w = v^3 = 1$. Hence

$$\tilde{C}_P : Z^6 + \alpha_3 \alpha_1^3 (X + Y)^3 Z^3 + \alpha_1 (X + Y) (\beta_1 X + \beta_2 Y) (\alpha_1^4 (X + Y)^4 + (\beta_1 X + \beta_2 Y)^4).$$

Consequently, $\beta_1 = \beta_2$ (because we are assuming $[Y; X; vZ] \in Aut(\tilde{C}_P)$), a contradiction to invertibility of P . Finally, if $[X, \xi_6^r Y, \xi_6^{r'} Z] \in Aut(\tilde{C}_P)$ then $r = 0$ and $2|r'$, since $\alpha_1 \alpha_2 \neq 0$. That is, $|Aut(\tilde{C}_P)| = 3$, which was to be shown. \square

As a conclusion of the results that are introduced in this section, we get the following result.

Corollary 5.5. *The locus $\widetilde{M_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})}$ is not ES-Irreducible, and it has at least two irreducible components.*

6. POSITIVE CHARACTERISTIC

Fix a prime $p > 0$ and let \mathbb{K} be an algebraically closed field of positive characteristic $p > 0$. Denote by $M_{g,p}$ the moduli space on smooth, genus g curves over the field \mathbb{K} , and similarly, we define the loci $M_{g,p}(G)$, $M_{g,p}^{Pl}(G)$ and $\widetilde{M_{g,p}^{Pl}(G)}$ over \mathbb{K} , as we did for zero characteristic.

Following the abuse of notation of §3, we consider a non-singular, degree d plane curve C in $\mathbb{P}^2(\mathbb{K})$, and assume that the order of $Aut(C)$ is coprime with p , $p \nmid d(d-1)$ and $p \geq 7$. Also, suppose that the order of $Aut(F_d)$ and $Aut(K_d)$ are coprime with p where $F_d : X^d + Y^d + Z^d = 0$ is the Fermat curve, and $K_d : X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X = 0$ is the Klein curve. Then, all techniques, which appeared in Harui [11], can be applied: Hurwitz bound, Arakawa and Oikawa inequalities and so on. In particular, the arguments of the previous sections hold.

Consider the p -torsion of the degree 0 Picard group of C , which is a finitely generated $\mathbb{Z}/(p)$ -module of dimension γ (always $\gamma \leq g$, where g is the genus of C). We call γ the p -rank of C .

For a point \mathcal{P} of C , denote by $Aut(C)_{\mathcal{P}}$ the subgroup of $Aut(C)$ that fixes the place \mathcal{P} .

Lemma 6.1. *Assume that $Aut(C)_{\mathcal{P}}$ is prime to p for any point \mathcal{P} of C and the p -rank of C is trivial. Then $Aut(C)$ is prime to p .*

Proof. Let $\sigma \in \text{Aut}(C)$ be of order p . Then the extension $\mathbb{K}(C)/\mathbb{K}(C)^\sigma$ is a finite extension of degree p , and is unramified everywhere (because if it ramifies at a place \mathcal{P} then σ will be an element of $\text{Aut}(C)_{\mathcal{P}}$ giving a contradiction). But, if $\gamma = 0$ (i.e. the p -rank is trivial for C) then, from Deuring-Shafarevich formula [13, Theorem 11.62], we get $\frac{\gamma-1}{\gamma'-1} = p$ where γ' is the p -rank for $C/\langle \sigma \rangle$, which is impossible. Therefore, such extensions do not exist. \square

Lemma 6.2. *Let C be a plane non-singular curve of degree $d \geq 4$. If $p > (d-1)(d-2) + 1$, then $\text{Aut}(C)_{\mathcal{P}}$ is coprime with p for any point \mathcal{P} of the curve C .*

Proof. By [13, Theorem 11.78], the maximal order of the p -subgroup of $\text{Aut}(C)_{\mathcal{P}}$ is at most $\frac{4p}{(p-1)^2}g^2$. Hence, with $g = \frac{(d-1)(d-2)}{2}$ and assuming that $p > \frac{4p}{(p-1)^2}g^2$, we obtain the result. \square

Lemma 6.3. *Let C be a non-singular curve of genus $g \geq 2$ that is defined over an algebraic closed field \mathbb{K} of characteristic $p > 0$. Suppose that C has an unramified subcover of degree p , i.e. $\Phi : C \rightarrow C'$ of degree p . Then, C' has genus ≥ 2 , $g \equiv 1 \pmod{p}$ and $\gamma \equiv 1 \pmod{p}$. In particular, one needs to assume that $p < g$ for the existence of such subcover.*

Proof. The Hurwitz formula for Φ gives the equality $(2g-2) = p(2g'-2)$, where g' is the genus of C' . We have $g' \neq 0$ or 1 because $g \geq 2$, therefore $g' \geq 2$ and $g-1 \equiv 0 \pmod{p}$. Now, consider the Deuring-Shafarevich formula, which could be read as $\gamma-1 = p(\gamma'-1)$ in such unramified extension, where γ' the p -rank of C' . If $\gamma = 1$ then there is nothing to prove and if $\gamma > 1$ then the congruence is clear. Finally, the situation $\gamma = 0$ does not occur. \square

Corollary 6.4. *Let C be a non-singular, degree d plane curve of genus $g \geq 2$ over an algebraic closed field \mathbb{K} of characteristic $p > 0$. Suppose that $p > (d-1)(d-2) + 1 > g$, then the order of $\text{Aut}(C)$ is coprime with p .*

Proof. Suppose $\sigma \in \text{Aut}(C)$ of order p , then $\mathbb{K}(C)/\mathbb{K}(C)^\sigma$ is a separable degree p extension, and by Lemma 6.2, it is unramified everywhere. By Lemma 6.3, we conclude that such extensions do not exist. \square

As a direct consequence of the above lemmas, and because all techniques of [11] are applicable when $\text{Aut}(C)$ is coprime with p , then we obtain:

Corollary 6.5. *Assume that $p > 13$, then the automorphism groups of the curves $\tilde{C} : X^5 + Y^5 + Z^4X + \beta X^3Y^2 = 0$ and $\tilde{\tilde{C}} : X^5 + X(Z^4 + Y^4) + \beta Y^2Z^3 = 0$ such that $\beta \neq 0$, are cyclic of order 4. Moreover, \tilde{C} is not isomorphic to $\tilde{\tilde{C}}$ for any choice of the parameters.*

Proof. We need only to mention that the linear g_2 -systems for the immersion of the curve inside \mathbb{P}^2 are unique, up to conjugation in $PGL_3(\mathbb{K})$ (see [13, Lemma 11.28]). Also, the curves \tilde{C} and $\tilde{\tilde{C}}$ have cyclic covers of degree 4 with different type of the cover, from Hurwitz equation. Therefore, they belong to different irreducible components in the moduli space of genus 6 curves. \square

Corollary 6.6. *The locus $M_{6,p}^{Pl}(\widetilde{\mathbb{Z}/4\mathbb{Z}})$ with $p > 13$ has at least two irreducible components.*

Similarly, we get the following results from those of §4 and §5,

Corollary 6.7. *The locus $M_{g,p}^{Pl}(\widetilde{\mathbb{Z}/(d-1)\mathbb{Z}})$ is not ES-Irreducible, and it has at least two strongly equation components for any odd integer $d \geq 5$ such that $p > (d-1)(d-2) + 1$. In particular, it has at least two irreducible components.*

Corollary 6.8. *For $p > 21$, the locus $M_{10,p}^{Pl}(\widetilde{\mathbb{Z}/3})$ has at least two irreducible components.*

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